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ONE NUMERICAL METHOD TO CALCULATE OPTIMAL POWER-LIMITED TRAJECTORIES

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Abstract

There are conventionally used different kinds of the Newton's methods and the shooting ones to solve boundary value problems of the celestial mechanics and space flight mechanics. These classes of methods have a number of deficiencies. Small dimension of the convergence region is the most essential deficiency generally. This causes great difficulties to choose initial approximation for solving boundary value problems.

It is considered the possibility to use method of differentiating with respect to parameter for constructing algorithm of solving of two point boundary value problem (TPBVP) to optimize power-limited transfers. The numerical algorithm of this method is described and examples of solutions are presented.

Nomenclature

CEV = constant ejection velocity model
LP = power limited model
TPBVP = two point boundary value problem
$\mathbf{x}$ = vector of the spacecraft's position
$\mathbf{v}$ = vector of the spacecraft's velocity
$\mathbf{a}$ = vector of the thrust acceleration
$\mathbf{p}_v$ = adjoint vector
t = time
T = transfer duration
Ω = force function of the gravity field
J = performance index
V_∞ = initial asymptotic geocentric velocity of the spacecraft
$\mathbf{z}$ = vector of the unknown TPBVP's parameters
τ = continuation parameter

Introduction

Two mathematical propulsion models have most frequently been used[1]. In the constant ejection velocity (CEV) model, either the thrust or thrust acceleration is bounded. In the power-limited (LP) model, power of the jet (the product of the thrust magnitude and the specific impulse) is bounded. This paper is concerned to investigation of the LP model only. The relation between these two models and possibility to use LP-solution to solve CEV-problem is considered in another our paper[2].

It is assumed that spacecraft moves in the force field Ω. Equation of spacecraft's motion in the inertial Cartesian coordinates is follows:

$$d^2\mathbf{x}/dt^2 = \Omega_{\mathbf{x}} + \mathbf{a}, \tag{1}$$

where $\mathbf{x}=(x, y, z)^T$ - vector of the spacecraft's position, $\mathbf{a}$ - vector of the thrust acceleration. The minimum-fuel transfer problem is reduced to the minimization of the performance index[1]

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$$J = \frac{1}{2}\int_0^T \mathbf{a}^{\mathrm{T}}\mathbf{a}\, dt, \tag{2}$$

where T is transfer duration. It is assumed that transfer duration is fixed. Optimal control problem is reduced to the TPBVP by means of Pontryagin's maximum principle. Optimal control is

$$\mathbf{a} = \mathbf{p}_{\mathrm{v}}, \tag{3}$$

where adjoint vector $\mathbf{p}_{\mathrm{v}}$ complies with the differential equation

$$\mathrm{d}^2\mathbf{p}_{\mathrm{v}}/\mathrm{d}t^2 = \Omega_{\mathbf{xx}}\, \mathbf{p}_{\mathrm{v}}. \tag{4}$$

Boundary and transversality conditions complete formulation of the TPBVP:

$$\mathbf{c}[\mathbf{x}(0),\mathrm{d}\mathbf{x}(0)/\mathrm{d}t,\ \mathbf{p}_{\mathrm{v}}(0),\mathrm{d}\mathbf{p}_{\mathrm{v}}(0)/\mathrm{d}t,\ \mathbf{x}(T),\mathrm{d}\mathbf{x}(T)/\mathrm{d}t,\ \mathbf{p}_{\mathrm{v}}(T),\mathrm{d}\mathbf{p}_{\mathrm{v}}(T)/\mathrm{d}t,T] = 0. \tag{5}$$

For example, let us consider optimization of the heliocentric arc of the interplanetary mission. Boundary conditions at the t=0 are[3]

$$\mathbf{x}(0) = \mathbf{x}_{\mathrm{o}},\ \mathrm{d}\mathbf{x}(0)/\mathrm{d}t = \mathbf{v}_{\mathrm{o}} + V_\infty\, \mathbf{p}_{\mathrm{v}}/p_v, \tag{6}$$

and ones at the $t = T$ are

$$\mathbf{x}(T) = \mathbf{x}_{\mathrm{k}},\ \mathrm{d}\mathbf{x}(T)/\mathrm{d}t = \mathbf{v}_{\mathrm{k}} \tag{7}$$

in case of rendezvous mission or

$$\mathbf{x}(T) = \mathbf{x}_{\mathrm{k}},\ \mathbf{p}_{\mathrm{v}}(T) = 0 \tag{8}$$

in case of flyby mission, where $p_v = |\mathbf{p}_{\mathrm{v}}|$ and V_∞ is initial asymptotic geocentric velocity of the spacecraft.

Continuation method

It is necessary to solve equation $\mathbf{f}(\mathbf{z}) = 0$, where $\mathbf{z}$ is unknown parameters of the TPBVP ($\mathbf{z}$ = $(\mathbf{p}_{\mathrm{v}}(0),\ \mathrm{d}\mathbf{p}_{\mathrm{v}}(0)/\mathrm{d}t)^{\mathrm{T}}$ = $(\mathbf{p}_{\mathrm{vo}}, \dot{\mathbf{p}}_{\mathrm{vo}})^{\mathrm{T}}$ in the considered case). Let us suppose that some initial approximation (may be zero one) of the TPBVP's parameter is known: $\mathbf{z}$=$\mathbf{z}_{\mathrm{o}}$, and $\mathbf{f}(\mathbf{z}_0) = \mathbf{b}$. Let us suppose that $\mathbf{z}$ depends on an artificial parameter τ, and eq. $\mathbf{f}(\mathbf{z}_0) = \mathbf{b}$ is correct when τ =0 and eq. $\mathbf{f}(\mathbf{z}) = 0$ is correct when τ=1. This statement can be written in form $\mathbf{f}(\mathbf{z}) = (1-\tau)\mathbf{b}$, $\mathbf{z}|_{\tau=0} = \mathbf{z}_0$, $\mathbf{z}|_{\tau=1} = \tilde{\mathbf{z}}$. It is obviously that $\tilde{\mathbf{z}}$ is solution of the TPBVP. Differentiation of this equation with respect to τ led to

$$\frac{d\mathbf{z}}{d\tau} = -\mathbf{f}_{\mathbf{z}}^{-1}(\mathbf{z})\mathbf{b},\quad \mathbf{z}(0) = \mathbf{z}_0 \,. \tag{9}$$

We will obtain solution of the TPBVP if we would integrate this equation from τ=0 to τ=1. It is of interesting that this equation led to the Newton's method when this system is integrated by Euler's method. Great potential possibilities of the considered method of the differentiating with respect to parameter are contained in the opportunity to use more precise integration method to integrate this system. So, author tested adaptive integration methods of Runge-Kutta of the 8-th order (algorithm DOPRI8[4]) and Everhart's method[5] of 15-th order (algorithm RA15). Use of these methods means use of derivatives of TPBVP's parameters with respect to τ up to 8-th or 15-th order respectively in the solving process. Namely this explains essential expansion of the convergence region in comparison with the Newton's method, which uses only the first-order derivatives. Computation of the Jacobian $\mathbf{f}_{\mathbf{z}}$ remains essential problem. Use of numerical differentiating led to computational instability when it is solving TPBVP. Therefore, it is necessary simultaneously to compute phase vector and Jacobian when dynamical system is integrated. Complete system of the differential equations of this inner problem has follows explicit form:

$$
\left.
\begin{aligned}
&\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = \Omega_{\mathrm{x}} + \mathbf{p}_{\mathrm{v}},\\
&\frac{\mathrm{d}^2\mathbf{p}_{\mathrm{v}}}{\mathrm{d}t^2} = \Omega_{\mathrm{xx}}\mathbf{p}_{\mathrm{v}},\\
&\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial\mathbf{x}}{\partial\mathbf{p}_{\mathrm{vo}}}\right) = \Omega_{\mathrm{xx}}\frac{\partial\mathbf{x}}{\partial\mathbf{p}_{\mathrm{vo}}} + \frac{\partial\mathbf{p}_{\mathrm{v}}}{\partial\mathbf{p}_{\mathrm{vo}}},\\
&\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial\mathbf{p}_{\mathrm{v}}}{\partial\mathbf{p}_{\mathrm{vo}}}\right) = \frac{\partial}{\partial\mathbf{x}}\left(\Omega_{\mathrm{xx}}\mathbf{p}_{\mathrm{v}}\right)\frac{\partial\mathbf{x}}{\partial\mathbf{p}_{\mathrm{vo}}} + \Omega_{\mathrm{xx}}\frac{\partial\mathbf{p}_{\mathrm{v}}}{\partial\mathbf{p}_{\mathrm{vo}}},\\
&\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial\mathbf{x}}{\partial\dot{\mathbf{p}}_{\mathrm{vo}}}\right) = \Omega_{\mathrm{xx}}\frac{\partial\mathbf{x}}{\partial\dot{\mathbf{p}}_{\mathrm{vo}}} + \frac{\partial\mathbf{p}_{\mathrm{v}}}{\partial\dot{\mathbf{p}}_{\mathrm{vo}}},\\
&\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial\mathbf{p}_{\mathrm{v}}}{\partial\dot{\mathbf{p}}_{\mathrm{vo}}}\right) = \frac{\partial}{\partial\mathbf{x}}\left(\Omega_{\mathrm{xx}}\mathbf{p}_{\mathrm{v}}\right)\frac{\partial\mathbf{x}}{\partial\dot{\mathbf{p}}_{\mathrm{vo}}} + \Omega_{\mathrm{xx}}\frac{\partial\mathbf{p}_{\mathrm{v}}}{\partial\dot{\mathbf{p}}_{\mathrm{vo}}}.
\end{aligned}
\right\}
\tag{10}
$$

Initial conditions (6) should be expanded by relations

$$
\begin{gathered}
\frac{\partial\mathbf{x}(0)}{\partial\mathbf{p}_{\mathrm{vo}}} = \frac{\partial\mathbf{x}(0)}{\partial\mathbf{p}_{\mathrm{vo}}} = \frac{\partial\mathbf{p}_{\mathrm{v}}(0)}{\partial\dot{\mathbf{p}}_{\mathrm{vo}}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathbf{p}_{\mathrm{v}}(0)}{\partial\mathbf{p}_{\mathrm{vo}}}\right) = 0,\\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathbf{x}(0)}{\partial\mathbf{p}_{\mathrm{vo}}}\right) = \frac{V_\infty}{p_{vo}}\left(\mathbf{E} - \frac{1}{p_{vo}^2}\mathbf{p}_{\mathrm{vo}}\mathbf{p}_{\mathrm{vo}}^{\mathrm{T}}\right),\ \frac{\partial\mathbf{p}_{\mathrm{v}}(0)}{\partial\mathbf{p}_{\mathrm{vo}}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathbf{p}_{\mathrm{v}}(0)}{\partial\dot{\mathbf{p}}_{\mathrm{vo}}}\right) = \mathbf{E},
\end{gathered}
\tag{11}
$$

where $\mathbf{E}$ is identity matrix, and $p_{vo} = |\mathbf{p}_{\mathrm{vo}}|$.

Vector $\mathbf{f}$ in (9) is defined by equation $\mathbf{f} = \begin{pmatrix} \mathbf{x}(T) - \mathbf{x}_{\mathrm{k}} \\ \mathrm{d}\mathbf{x}(T)/\mathrm{d}t - \mathbf{v}_{\mathrm{k}} \end{pmatrix}$ in case of rendezvous or $\mathbf{f} = \begin{pmatrix} \mathbf{x}(T) - \mathbf{x}_{\mathrm{k}} \\ \mathbf{p}_{\mathrm{v}}(T) \end{pmatrix}$ in case of flyby mission. Therefore, Jacobian $\mathbf{f_z}$ in (9) has form

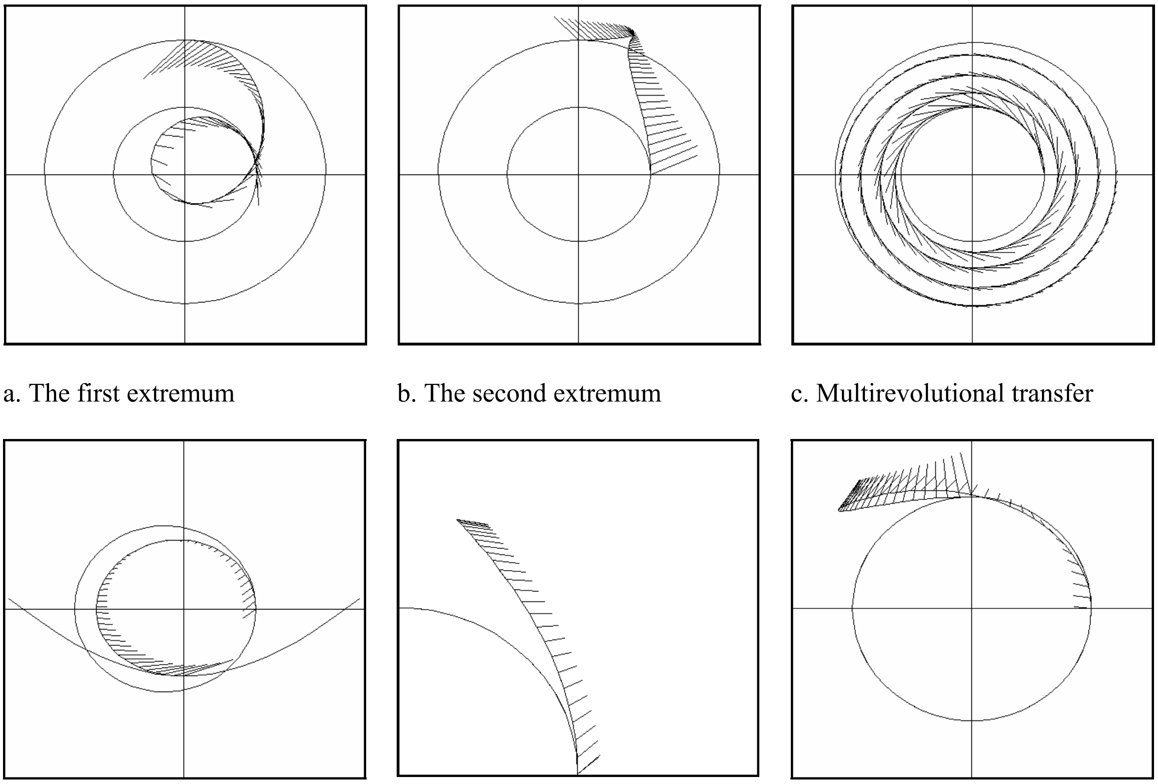
$\mathbf{f}_{\mathrm{z}} = \begin{pmatrix} \partial\mathbf{x}(T)/\partial\mathbf{p}_{\mathrm{vo}} & \partial\mathbf{x}(T)/\partial\dot{\mathbf{p}}_{\mathrm{vo}} \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\partial\mathbf{x}(T)/\partial\mathbf{p}_{\mathrm{vo}}\right) & \frac{\mathrm{d}}{\mathrm{d}t}\left(\partial\mathbf{x}(T)/\partial\dot{\mathbf{p}}_{\mathrm{vo}}\right) \end{pmatrix}$ or $\mathbf{f}_{\mathrm{z}} = \begin{pmatrix} \partial\mathbf{x}(T)/\partial\mathbf{p}_{\mathrm{vo}} & \partial\mathbf{x}(T)/\partial\dot{\mathbf{p}}_{\mathrm{vo}} \\ \partial\mathbf{p}_{\mathrm{v}}(T)/\partial\mathbf{p}_{\mathrm{vo}} & \partial\mathbf{p}_{\mathrm{v}}(T)/\partial\dot{\mathbf{p}}_{\mathrm{vo}} \end{pmatrix}$ in cases of rendezvous or flyby missions respectively.

There were tested algorithms RA15 and high-precise extrapolation one ODEX2[4] to integrate this inner problem. Use of DOPRI8 in outer problem (9) and ODEX2 in inner one ((10,6,11) and (7) or (8)) was found optimal from point of view computational productivity.

Numerical examples

Algorithm, which was worked out, was implemented as part of the mission analysis and design software ITCAD 2.0 (Interplanetary transfers CAD) for personal computers IBM PC/AT. There were calculated a number of the trajectories by means of this toolkit, including missions to the planets, asteroids and user-defined objects. There are some examples of these optimal trajectories on the Fig. 1. Strokes on the trajectories denote direction and magnitude of the thrust acceleration. It is important to note that all these solutions were obtained from the zero initial approximation of the TPBVP's parameters: $\mathbf{p}_{\mathrm{vo}}\big|_{\tau=0} = \dot{\mathbf{p}}_{\mathrm{vo}}\big|_{\tau=0} = 0$.

First two figures (Fig.1a and Fig.1b) present two different extremums of the same TPBVP. Example of the multirevolutional transfer is shown on the Fig.1c. Fig.1d presents solution of the escape problem. Last two figures (Fig.1e and Fig.1f) present exotic trajectories in which direction of motion is changed.



a. The first extremum | b. The second extremum | c. Multirevolutional transfer

d. Ellipse-to-hyperbola transfer | e. Change of motion direction | f. Case e) and phase change

Fig.1. Examples of the optimal transfers

The next examples are more realistic. It is considered Pluto and Mercury rendezvous missions. It is assumed that spacecraft is delivered into the hyperbolic geocentric orbit by Russian launcher “Proton” with upper stage “Block D”. It is assumed that electric propulsion thrusters run in the heliocentric arc of the trajectory only. The external parameters of the TPBVP (launch date t_0 and V_∞) is optimized. Transfer duration and power limit are fixed. Some results of the optimization are presented in the Table 1 and Fig.2-4.

Table 1

Optimal transfers to the Pluto and Mercury

Parameter	Pluto rendezvous mission	Mercury rendezvous mission
Launch date	10.01.2000	15.08.2001
Transfer duration [d]	4500	1600
$\mathbf{x}(0)$ [a.u.]	(-0.32088, 0.92956, 0)	(0.80085, -0.62004, 0)
$\mathrm{d}\mathbf{x}(0)/\mathrm{d}t$ [a.u./d]	(-0.016543, -0.005678,0)	(0.010252, 0.013540, 0)
$\mathbf{x}(T)$ [a.u.]	(2.83018, -31.93666, 2.37474)	(-0.28105, -0.35790, 0.00337)
$\mathrm{d}\mathbf{x}(T)/\mathrm{d}t$ [a.u./d]	(0.003164, -0.000343, -0.000889)	(0.016426, -0.016072, -0.002820)
$\mathbf{p}_\mathbf{v}(0)$ [mm/s^2]	(-0.192371, -0.065533, 0.005084)	(0.358371, -0.138241, -0.115369)
$\mathrm{d}\mathbf{p}_\mathbf{v}(0)/\mathrm{d}t\times10^3$ [(mm/s^2)/d]	(0.960650, -2.787335, 0.202574)	(3.06483, -1.60851, 1.59627)
$2J$ [m^2/s^3]	5.58456	2.99175
Final spacecraft’s mass [kg]	3770	4503
Initial spacecraft’s mass [kg]	5807	
V_∞ [m/s]	1800	
Jet power [kW]	30	

Fig.2 presents Pluto rendezvous trajectory and dependency of the thrust acceleration with respect to time. Strokes on the trajectory denote magnitude and direction of the thrust acceleration. Time interval between adjacent strokes is 30 days. Mercury rendezvous trajectory is shown in the Fig.3 (interval between adjacent strokes is 4 days). It is chart of the corresponding thrust acceleration in the Fig.4.

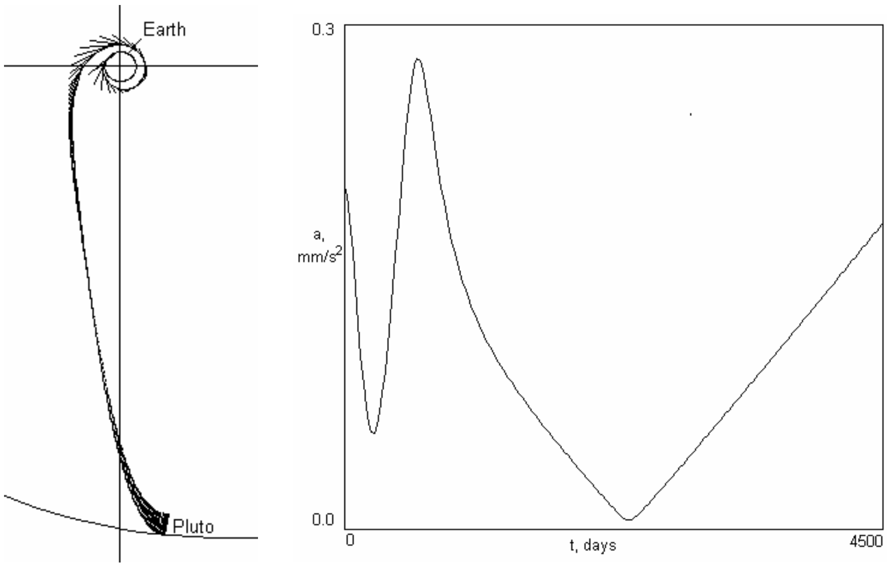


Fig.2. Pluto rendezvous trajectory and corresponding thrust acceleration.

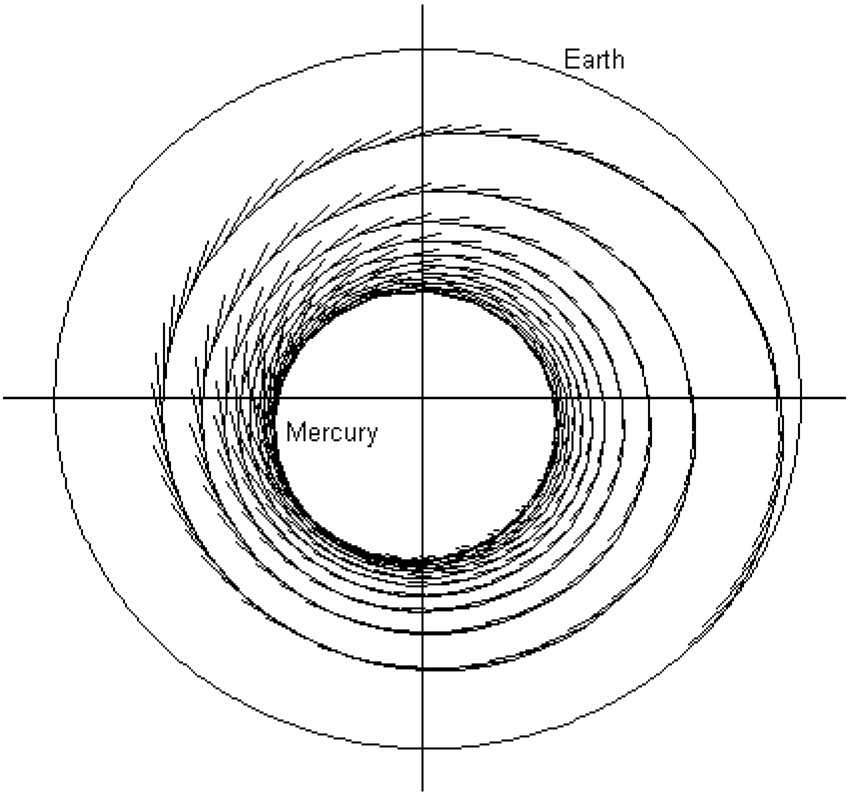


Fig.3. Mercury rendezvous trajectory.

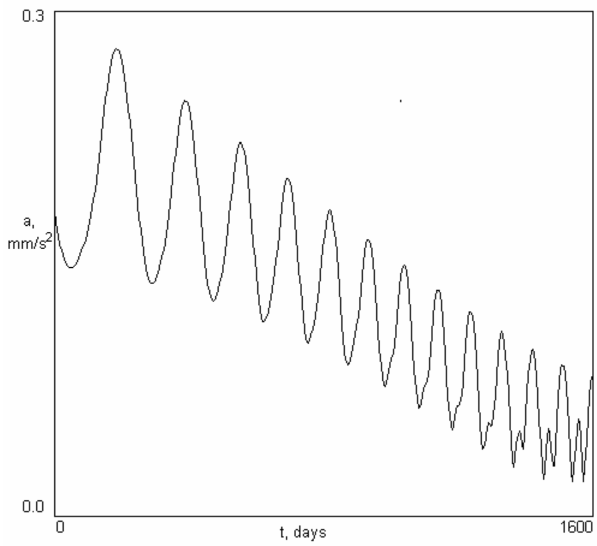


Fig.4. Thrust acceleration with respect to time (Mercury rendezvous).

The last example demonstrates nonuniqueness of the optimal solutions. It is considered optimization of the heliocentric arc of Pluto rendezvous mission. It is assumed that $V_\infty = 0$. Fig.5 presents three kinds of trajectories that is differed one from other by number of entire revolutions (0, 1, and 2). The corresponding dependencies of the thrust acceleration with respect to the time are presented in the Fig.6. Dependencies of the performance index J from the transfer duration T are presented in the Fig.7. As we can see, the first extremum is a global minimum up to T = 3760 days. The second kind of trajectories is prefered in case if T is within interval from 3760 to 5330 days, and the third one has a less perfomance index in case if T is greater than 5330 days.

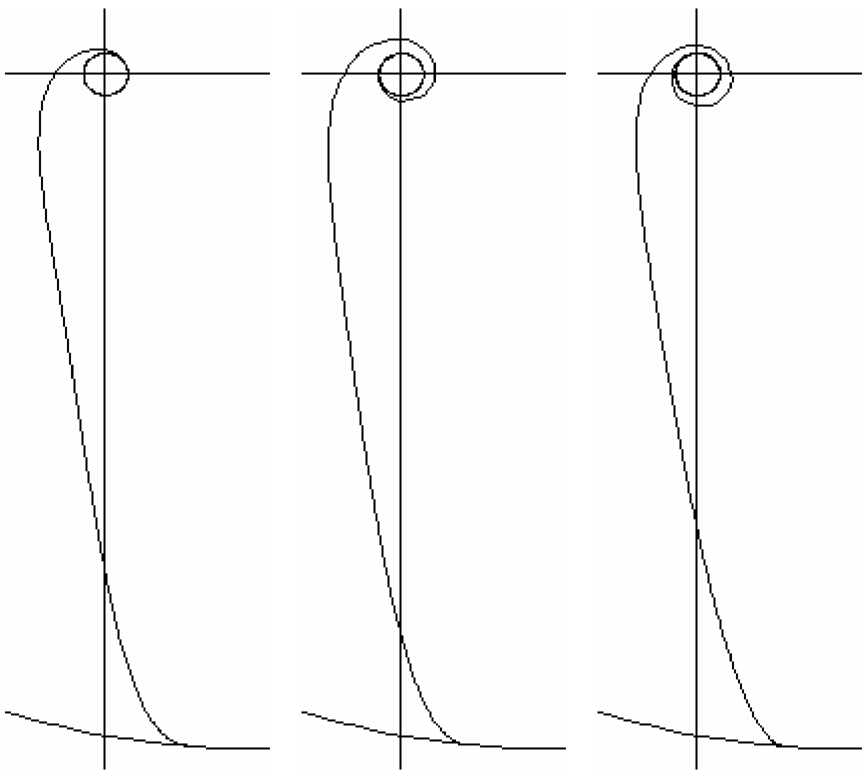


Fig.5. Different kinds of the optimal Pluto rendezvous trajectories.

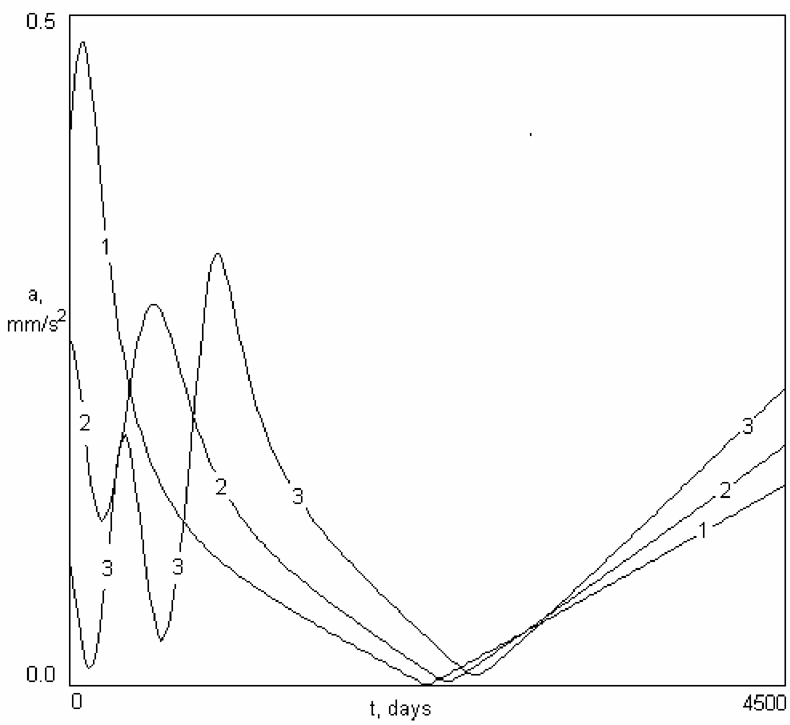


Fig.6. Thrust acceleration in case of different kind of Pluto rendezvous trajectory.

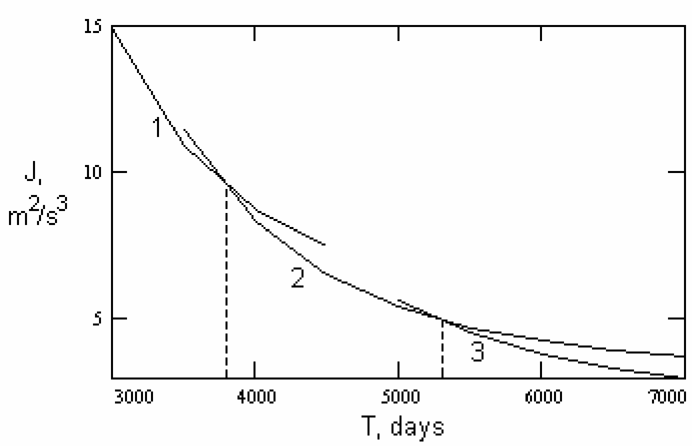


Fig.7. Perfomance index with respect to transfer time (three kinds of the Pluto rendezvous trajectories)

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